# Continuous Simultaneous Stabilization of Single-Input Nonlinear Stochastic Systems *

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**Abstract**

This paper investigates the simultaneous stabilization of a collection of continuous single-input non-linear stochastic systems, with coefficients that are not necessarily locally Lipschitz. A sufficient condition for the existence of a continuous simultaneously stabilizing feedback control is proposed — it is based on the generalized stochastic Lyapunov theorem and on the technique of stochastic control Lyapunov functions. This condition is also necessary, provided that the system's coefficients satisfy some regularity conditions. Moreover, the proposed feedback can be chosen to be bounded under the assumption that appropriate control Lyapunov functions are known. All the proposed simultaneously stabilizing state feedback controllers are explicitly constructed. Finally, two simulation examples are provided to demonstrate the effectiveness of the proposed approach.

**Keywords.** Nonlinear stochastic systems, continuous stochastic stabilization, simultaneous stabilization, control Lyapunov function, state feedback.

# 1 Introduction

Many real systems are nonlinear and are also prone to stochastic phenomena. These features naturally attracted the attention of the control literature to stabilization problems for stochastic nonlinear systems, and some interesting results have been obtained during over the past few decades— see [5, 11, 12, 16, 29, 36, 37] and references therein. These results were obtained using classical stochastic theory as found, for instance, in [18] and [26]. This theory requires the systems coefficients to be continuously differentiable, in order to guarantee the local Lipschitz condition.

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However, many stochastic models arising in applications do not satisfy such a local Lipschitz condition. For example, stochastic models for a sequencing-batch reactor and for a chemostat (see [7]); some stochastic financial and biological models (see [22, 27]). These models have coefficients involving the term $\sqrt{x(t)}$. In order to relax this restriction, Li & Liu [23] generalized the classical theorems on global stability to cover stochastic nonlinear systems without a local Lipschitz condition. Using the results in [23], several authors have obtained sufficient stabilizability conditions — e.g. [9, 29] — allowing closed-loop systems to be merely continuous without requiring them to satisfy a local Lipschitz condition. It should be pointed out that all results mentioned above are about stability and stabilization in probability problems. Indeed, there are fewer results on the simultaneous stabilization of stochastic nonlinear systems, due to the difficulties involved in synthesis.

As it is well known, simultaneous stabilization problems arise due to several reasons: system's uncertainty, failure modes or systems with various modes of operation. The aim of this problem is to determine a single controller which simultaneously stabilizes a finite collection of systems. Current applications of simultaneous stabilization include many real systems, such as Hamiltonian systems [1, 31], power converters [2], chemical reaction systems [38, 39], and descriptor systems [3] to name a few.

Therefore, the investigation of simultaneous stabilization problems is quite relevant from a practical point of view, and many important results have been obtained for linear systems: see [8, 10, 20, 24, 34] and references therein. For nonlinear systems, the problem is more involved, and there are fewer results in the literature — e.g. [4, 14, 32, 33, 35]. All these works deal with the simultaneous stabilization of linear and nonlinear deterministic systems.

To the authors' best knowledge, there are even fewer results on simultaneous stabilization of stochastic nonlinear systems — an exception being [13]. The goal of this paper is to contribute to this problem and to investigate the simultaneous stabilization in probability of a collection of single-input nonlinear stochastic systems, with a drift that is affine in the control. The authors showed in [13] that the simultaneous stabilization problem of a collection of single-input nonlinear stochastic systems, $\{S_i\}_{i=1}^m$, can be reduced to a simpler one, provided the following condition holds: for each system $S_i$ there exists a stochastic control Lyapunov function. In this case, the problem is then reduced to finding a single continuous feedback $u$ satisfying that for all $x \in \mathbb{R}^n \setminus \{0\}$, one has $a_i(x) + b_i(x)u(x) < 0$, with known functions, $a_i$ and $b_i$, $i = 1, 2, ..., m$. Under the additional assumption that, for each $x \in \mathbb{R}^n$, $\{b_i\}_{i=1}^m$ is single signed, an explicit feedback is constructed in [13].

The aim of this paper is to present a new sufficient condition for simultaneous stabilizability that improves the condition derived in [13]. This new condition will allow the removal of the aditional single signed condition on $\{b_i\}_{i=1}^m$, in order to be able to design simultaneous stabilizers for a wider class of a collection of stochastic differential systems. Moreover, this condition is also necessary, if the corresponding closed-loop systems satisfy some additional regularity condition, such as a Lipschitz condition. The main tools used in this work are the generalised stochastic Lyapunov theorem proved by Li & Liu [23], the technique of stochastic control Lyapunov function and the converse stability theorem of Kushner [21].

The remainder of this paper is organized as follows: Section 2 gives some preliminary material on asymptotic stability in probability of weak solutions. Section 3 formulates the stochastic nonlinear systems and the control design problem. Section 4 presents the design scheme of the state-feedback controller, and gives the main results of the paper. Finally, in section 5, we apply the results obtained in Section 4 to two examples that cannot be stabilized using the results available in [13].

## 2 Stochastic stability of weak solutions

The purpose of this section is to recall some basic concepts and results concerning the stability and asymptotic stability in probability of an equilibrium solution of a stochastic differential equation that will be used in the sequel. Consider the following stochastic nonlinear system:

$$dx = f(x)dt + h(x)d\omega, \quad x(0) = x_0 \in \mathbb{R}^n, \tag{2.1}$$

where $x \in \mathbb{R}^n$ is the system state; $\omega$ is an m-dimensional independent standard Wiener process. The function $f : \mathbb{R}^n \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are continuous and satisfy $f(0) = 0$ and $h(0) = 0$. Clearly, the origin is an equilibrium point of system (2.1).
Usually, in order to guarantee the existence and uniqueness of strong solutions of the stochastic systems (2.1), the functions $f(.)$ and $h(.)$ are assumed to satisfy some regularity conditions, such as local Lipschitz condition, (see Khas'minskii [18] and Mao [26]). In this paper, $f(.)$ and $h(.)$ are assumed to be continuous, but not locally Lipschtiz. It then follows that system (2.1) may not have solutions in the classical sense, as in Khas'minskii [18] and Mao [26]. However, system (2.1) always has weak solutions which are essentially different from the classical (or strong) solution since the former may not be pathwise unique and may be defined on a different probability space. The following definition of weak solution of the stochastic system (2.1) is given in [23], and for more details on this subject, we refer the reader to Ikeda and Watanabe [15], Klebaner [19], Ondreját and Seidler [28].

**Definition 2.1.** *[23]: If there exists a continuous adapted process $x(t)$ on a probability space $(\Omega^x, \mathcal{F}^x, P^x)$ with a filtration $\{\mathcal{F}_t\}_{t\geq t_0}$ satisfying the usual conditions, and an m-dimensional $\{\mathcal{F}_t\}$-adapted Brownian motion $\omega^x(t)$ with $P^x\{\omega^x(t_0) = 0\} = 1$, such that the initial condition $x(t_0)$ has the given distribution, and for all $t \in [t_0, \tau^x_{+\infty})$*

$$x(t) = x(t_0) + \int_{t_0}^{t} f(x(s))ds + \int_{t_0}^{t} h(x(s))d\omega^x(s) \quad a.s.,$$

*then $x(t)$ is called a weak solution of system (2.1), where $\tau^x_{+\infty}$ is the explosion time of the weak solution $x(t)$, that is $\tau^x_{+\infty} = \lim\limits_{\varepsilon\to+\infty} inf\{t \geq t_0, \|x(t)\| \geq \varepsilon, \forall\varepsilon > 0\}$.*

Given a twice-differentiable function $\Psi : \mathbb{R}^n \to \mathbb{R}$, the infinitesimal generator $\mathcal{L}$ associated with the stochastic differential equation (2.1) for $\Psi(x)$, is given by

$$\mathcal{L}\Psi(x) = \nabla\Psi(x)f(x) + \frac{1}{2}\mathrm{Tr}\Big(h(x)h(x)^\top\nabla^2\Psi(x)\Big),$$

where Tr$\{.\}$ represents the trace of the argument.
We now recall the following version of the stochastic Lyapunov theorem stated in [23], that concerns stochastic systems with coefficients that are only continuous.

**Lemma 2.1.** *[23]: For system (2.1), if there exists a $C^2$ function $V : \mathbb{R}^n \to \mathbb{R}^+$ and two $\mathcal{K}_\infty$ class functions $\alpha$ and $\beta$, such that*

$$\alpha(\|x\|) \leq V(x) \leq \beta(\|x\|),\ \mathcal{L}V(x) \leq -W(x),$$

*where $W(\cdot)$ is a nonnegative continuous function. Then the zero solution of system (2.1) is globally stable in probability, and for $\forall x_0 \in \mathbb{R}^n$, every weak solution $x(t)$ of system (2.1) satisfies*

$$P^x\{\lim_{t\to+\infty} W(x(t)) = 0\} = 1.$$

*Furthermore, if $W(\cdot)$ is positive definite, then the zero solution of system (2.1) is globally asymptotically stable in probability.*

# 3 Problem formulation and preliminaries

## 3.1 Problem formulation

Consider a collection of affine stochastic systems described by

$$S_i : dx = f_i(x)dt + ug_i(x)dt + h_i(x)d\omega, \quad i \in \{1, ..., m\} = I, \tag{3.1}$$

where $x \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}$ is the control and $\omega$ is a standard $\mathbb{R}^m$-valued Wiener process defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ with $\Omega$ being a sample space, $\mathcal{F}$ a $\sigma$-algebra on $\Omega$, $\mathcal{F}_t$ a filtration and $P$ a probability measure. $f_i$, $g_i : \mathbb{R}^n \to \mathbb{R}^n$ and $h_i : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are continuous functions with $f_i(0) = h_i(0) = 0$.
We deal with the design of a single feedback law $p : \mathbb{R}^n \to \mathbb{R}$ by explicit formulas in such a way that all resulting closed-loop systems

$$dx = f_i(x)dt + p(x)g_i(x)dt + h_i(x)d\omega, \quad i \in I$$

have the origin globally asymptotically stable in probability.

## 3.2 Stochastic control Lyapunov function

In the following, we present the concept of stochastic control Lyapunov function (SCLF), which plays an important role in our approach to solve the simultaneous stabilization problem. Consider a single nonlinear stochastic systems

$$dx = f(x)dt + ug(x)dt + h(x)d\omega. \tag{3.2}$$

For a $C^2$ function $V$, the infinitesimal operator $\mathcal{L}_u$ associated with the system (3.2) is defined for $V$ as $\mathcal{L}_u V(x) = a(x) + b(x)u$, with

$$\begin{aligned} a(x) &= \nabla V(x)f(x) + \tfrac{1}{2}\mathrm{Tr}\Big(h(x)h(x)^\top \nabla^2 V(x)\Big) \\ b(x) &= \nabla V(x)g(x). \end{aligned} \tag{3.3}$$

The following definition is required for stating the results of this paper.

**Definition 3.1.** *([6, 11]). A positive definite and proper function $V \in C^2(\mathbb{R}^n, \mathbb{R}^+)$ is said: 1) to be a stochastic control Lyapunov function (SCLF) for the stochastic differential system (3.2), if*

$$\inf_{u\in\mathbb{R}} \mathcal{L}_u V(x) = \inf_{u\in\mathbb{R}} \Big(a(x) + b(x)u\Big) < 0, \quad \forall x \in \mathbb{R}^n \setminus \{0\}. \tag{3.4}$$

*2) to satisfy the small control property with system (3.2), if for each $\varepsilon > 0$ there is $\delta > 0$ such that , if $x \neq 0$ satisfies $\|x\| < \delta$, then there is some $u$ with $\|u\| < \varepsilon$ such that*

$$\mathcal{L}_u V(x) = a(x) + b(x)u < 0.$$

Note that $V$ is a SCLF for the system (3.2) is equivalent to

$$(b(x) = 0, \quad x \neq 0) \quad \Rightarrow \quad a(x) < 0, \quad \forall x \in \mathbb{R}^n \setminus \{0\}. \tag{3.5}$$

# 4 Main results: Design method of stabilizing controller

This section presents a constructive method of a simultaneously stabilizing feedback, based on a stochastic control Lyapunov function approach. First, we recall the continuous feedback stabilization result. Then, we present a design method.

## 4.1 Continuous feedback stabilization

Let (as motivated by Sontag's universal formula [30])

$$k(x) = \begin{cases} -\dfrac{a(x) + \sqrt{a^2(x) + b^4(x)}}{b(x)}, & \text{if } \ b(x) \neq 0 \\ 0, & \text{if } \ b(x) = 0, \end{cases} \tag{4.1}$$

the authors showed in [13] the following theorem.

**Theorem 4.1.** *[13]: If there exists a stochastic control Lyapunov function $V$ for the system (3.2), satisfying the small control property, then the feedback $u = k(x)$, defined in (4.1), is continuous on $\mathbb{R}^n$ and globally asymptotically stabilizes in probability system (3.2).*

*Proof.* First, notice that the functions $a$ and $b$ are continuous, since $V \in C^2(\mathbb{R}^n, \mathbb{R}^+)$ and $f$, $g$ are continuous. Together with (3.5), the continuity of the function $k$ on $\mathbb{R}^n \setminus \{0\}$ is easy to verify. In addition, since $V$ satisfies the small control property, arguing as in [30], we can prove that $k$ is continuous on $\mathbb{R}^n$. It follows that the coefficients of the closed-loop system deduced from the stochastic system (3.2) with the control $u = k(x)$ are continuous on $\mathbb{R}^n$ as well as the associated infinitesimal operator $\mathcal{L}_k V$.
Second, for $x \in \mathbb{R}^n \setminus \{0\}$, if $b(x) \neq 0$ then $\mathcal{L}_k V(x) = a(x) + b(x)k(x) = -\sqrt{a^2(x) + b^4(x)} < 0$, and if $b(x) = 0$ then $\mathcal{L}_k V(x) = a(x) < 0$ since $V$ is a SCLF for system (3.2), that is $\mathcal{L}_k V$ is negative definite.
Now, since $V$ is a SCLF, that is $V$ is proper and positive definite, and $\mathcal{L}_k V$ is continuous and negative definite, according to Lemma 4.3 in [17], there exit class $\mathcal{K}_\infty$ functions $\alpha_1$ and $\alpha_2$ and class $\mathcal{K}$ function $\xi$, defined on $\mathbb{R}^+$, such that

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \quad \forall x \in \mathbb{R}^n,$$

$$\mathcal{L}_k V(x) = a(x) + b(x)k(x) \leq -\xi(\|x\|), \quad \forall x \in \mathbb{R}^n.$$

The conclusion follows with the help of Lemma 2.1. □

**Remark 4.1.** *In fact, another continuous asymptotically stabilizing controller can be found for system (3.2). If the assumptions of Theorem 4.1 hold, then the functions $a$ and $b$ are continuous, and since $V$ is a SCLF for system (3.2), that is $b(x) = 0$ $x \neq 0$ implies $a(x) < 0$, it is easy to verify that the feedback*

$$u(x) = \begin{cases} \min\left(0, -\dfrac{a(x)}{b(x)} - b(x)\right), & \text{if } \ b(x) > 0 \\ \max\left(0, -\dfrac{a(x)}{b(x)} - b(x)\right), & \text{if } \ b(x) < 0 \\ 0, & \text{if } \ b(x) = 0, \end{cases}$$

*is continuous and globally asymptotically stabilizes system (3.2).*

## 4.2 Simultaneous stabilization

As a first step towards the construction of a simultaneously stabilizing feedback law for the collection of stochastic systems (3.1), we introduce the following sets and functions: For each $i \in I$, suppose that $V_i$ is a SCLF for system $S_i$ which satisfies the small control property, and let ($x$ is dropped for space reason)

$$a_i = \nabla V_i f_i + \frac{1}{2}\mathrm{Tr}\Big(h_i h^\top \nabla^2 V_i\Big),\ b_i = \nabla V_i g_i, \tag{4.2}$$

and $k_i(x) = \begin{cases} -\dfrac{a_i(x) + \sqrt{a_i^2(x) + b_i^4(x)}}{b_i(x)}, & \text{if } \ b_i(x) \neq 0 \\ 0. & \text{if } \ b_i(x) = 0. \end{cases}$

As shown in Theorem 4.1, the functions $k_i$, $i = 1, 2, ..., m$ are continuous on $\mathbb{R}^n$. For each $x \in \mathbb{R}^n$, let

$$I_p(x) = \{i \in I, \quad b_i(x) > 0\},\ I_n(x) = \{i \in I, \quad b_i(x) < 0\},$$

and for each $i \in I$, define the functions $\varphi_{1i}$, $\varphi_{2i}$, $\psi_{1i}$ and $\psi_{2i}$ on $\mathbb{R}^n$ as follows:

$$\varphi_{1i}(x) = \begin{cases} -\dfrac{a_i(x)}{b_i(x)} & \text{if } \ b_i(x) < 0, \\ -\infty & \text{if } \ b_i(x) \geq 0, \end{cases}$$
$$\varphi_{2i}(x) = \begin{cases} -\dfrac{a_i(x)}{b_i(x)} & \text{if } \ b_i(x) > 0, \\ +\infty & \text{if } \ b_i(x) \leq 0, \end{cases}$$
$$\psi_{1i}(x) = k_i(x) \text{ if } b_i(x) < 0 \text{ and } 0 \text{ if } b_i(x) \geq 0,$$
$$\psi_{2i}(x) = k_i(x) \text{ if } b_i(x) > 0 \text{ and } 0 \text{ if } b_i(x) \leq 0.$$

Now, let $\varphi_1(x) = \max\limits_{i\in I} \varphi_{1i}(x), \qquad \varphi_2(x) = \min\limits_{i\in I} \varphi_{2i}(x),$

$$\psi_1(x) = \max_{i\in I} \psi_{1i}(x), \qquad \psi_2(x) = \min_{i\in I} \psi_{2i}(x).$$

Finally, define

$$w_1(x) = \min(\psi_1(x), \varphi_2(x)),\ w_2(x) = \max(\psi_2(x), \varphi_1(x)).$$

**Remark 4.2.** *Since $a_i(x) + \sqrt{a_i^2(x) + b_i^4(x)} > 0$, it follows that $b_i(x)k_i(x) < 0$, if $b_i(x) \neq 0$. So, $\psi_{1i}(x) = k_i(x) > 0$ if $b_i(x) < 0$, and since $\psi_{1i}(x) = 0$ if $b_i(x) \geq 0$, we get $0 \leq \psi_{1i}(x)$ for all $i \in I$ and for all $x \in \mathbb{R}^n$. Thus $0 \leq \psi_1(x), \quad \forall x \in \mathbb{R}^n$. $\psi_{2i}(x) = k_i(x) < 0$ if $b_i(x) > 0$, and since $\psi_{2i}(x) = 0$ if $b_i(x) \leq 0$, we get $\psi_{2i}(x) \leq 0$ for all $i \in I$ and for all $x \in \mathbb{R}^n$. Thus $\psi_2(x) \leq 0, \quad \forall x \in \mathbb{R}^n$.*

Now, we give our sufficient stabilizability condition and we state the main results of this paper.

**Assumption 4.1.** *For every $x \in \mathbb{R}^n \setminus \{0\}$, $\varphi_1(x) < \varphi_2(x)$.*

**Remark 4.3.** *Note that Assumption 4.1: $\varphi_1(x) < \varphi_2(x)$, $x \in \mathbb{R}^n$ is equivalent to: for all $x \in \mathbb{R}^n$ s. t. $I_n(x) \neq \emptyset$ and $I_p(x) \neq \emptyset$*

$$\max_{i\in I_n(x)} \Big(-\frac{a_i(x)}{b_i(x)}\Big) < \min_{i\in I_p(x)} \Big(-\frac{a_i(x)}{b_i(x)}\Big).$$

**Remark 4.4.** *Assumption 4.1 relaxes the stabilizability condition stated in [13], where all the term $b_i(x)$, $i \in I$ were supposed to have the same sign for each $x \in \mathbb{R}^n$. However Assumption 4.1 makes the feedback control design of the collection of systems in (3.1) more complicated and cannot be directly solved by the method in [13].*

Moreover, Assumption 4.1 becomes necessary under some regularity conditions. As it is well known in stochastic differential equation theory [18, 26], in order to guarantee the existence and uniqueness of strong solutions of an unforced stochastic systems, its coefficients are assumed to satisfy some definite conditions such as Lipschitz and linear growth conditions. Now, for stochastic control systems in (3.1), assume that for each $i \in I$, $f_i$, $g_i$ and $h_i$ satisfy the above definite conditions, we also assume that there exists a common control feedback $k$ that satisfies the above definite conditions and globally asymptotically stabilizes the collection in (3.1), clearly, the coefficients of the resulting closed-loop system $dx = f_i(x)dt + k(x)g_i(x)dt + h_i(x)d\omega$ satisfy the Lipschitz and linear growth conditions. Then we have the following result.

**Theorem 4.2.** *Consider the collection of systems in (3.1). If there exists a feedback $k : \mathbb{R}^n \to \mathbb{R}$ such that for each $i \in I$, the closed-loop system $dx = f_i(x)dt + k(x)g_i(x)dt + h_i(x)d\omega$, has its coefficients satisfying the above definite conditions and is globally asymptotically stable in probability, then there exists a collection of SCLFs $V_i$ for systems $S_i$, respectively, satisfying Assumption 4.1.*

*Proof.* Suppose that there exists a feedback $k$ such that the closed-loop systems $dx = f_i(x)dt + k(x)g_i(x)dt + h_i(x)d\omega$, $i \in I$, are globally asymptotically stable in probability, then by Kushner's converse Lyapunov theorem [21], there exist $C^2$ positive definite and radially unbounded functions $V_i$, $i \in I$, such that for each $i \in I$, $\mathcal{L}_u V_i(x) = a_i(x) + b_i(x)k(x) < 0$ , for all $x \neq 0$. It then follows that for each $x \in \mathbb{R}^n$, one has:
$k(x) < -\frac{a_i(x)}{b_i(x)}$ if $b_i(x) > 0$ $i \in I$, then $k(x) < \varphi_{2i}(x)$ for all $i \in I$ since $\varphi_{2i}(x) = +\infty$ if $b_i(x) \leq 0$, consequently, $k(x) < \varphi_2(x)$.
$k(x) > -\frac{a_i(x)}{b_i(x)}$ if $b_i(x) < 0$ $i \in I$, then $k(x) > \varphi_{1i}(x)$ for all $i \in I$ since $\varphi_{1i}(x) = -\infty$ if $b_i(x) \geq 0$, consequently, $k(x) > \varphi_1(x)$.
Thus, $\varphi_1(x) < \varphi_2(x)$, for all $x \in \mathbb{R}^n$ and this completes the proof of Theorem 4.2. □

**Remark 4.5.** *Note that under the assumptions in Theorem 4.2, the existing SCLFs $V_i$ of systems in (3.1) do not necessary satisfy the stabilizability condition stated in [13].*

Now, even though all the functions $f_i$, $g_i$ and $h_i$, $i \in I$, of the systems in (3.1) can be chosen as in Theorem 4.2, i.e satisfying the Lipschitz and linear growth conditions, and there exists a collection of SCLFs $V_i$, $i \in I$, of system $S_i$ $i \in I$, respectively, that satisfy Assumption 4.1, it is much more difficult to construct a control feedback satisfying the Lipschitz and linear growth conditions and that globally asymptotically stabilizes in probability the collection of systems in (3.1) simultaneously. But, as shown in the following result, we are able to design a common merely continuous stabilizing feedback of systems in (3.1).

**Theorem 4.3.** *Consider the collection of systems in (3.1). If there exists a SCLF $V_i$, that satisfies the small control property, for every $S_i$, $i \in \{1, ..., m\}$, and Assumption 4.1 holds, then the feedback*

$$u(x) = \frac{w_1(x) + w_2(x)}{2} \tag{4.3}$$

*is continuous and globally asymptotically stabilizes in probability the collection of systems in (3.1) simultaneously.*

*Proof.* **Continuity of $u$:** We first prove the continuity of $w_1$. Since $k_i$ is continuous and $k_i(x) = 0$ whenever $b_i(x) = 0$, it is easy to see that $\psi_{1i}$, $i \in I$, is continuous in $\mathbb{R}^n$, then so is $\psi_1$. Now let $x_0 \in \mathbb{R}^n \setminus \{0\}$, we will verify that there exists $\eta_{x_0} > 0$, such that we either have $w_1(x) = \psi_1(x)$ for all $x \in B(x_0, \eta_{x_0})$ if $I_p(x_0) = \emptyset$, or $\varphi_2$ is continuous on $B(x_0, \eta_{x_0})$ if $I_p(x_0) \neq \emptyset$. For each $i \in I$, from the definition of $\varphi_{2i}$ and the fact that $V_i$ is a SCLF for system $S_i$, it's not hard to verify that, if $b_i(x_0) \le 0$, then $\forall L > 0,\ \ \exists \lambda_L^i > 0$ s. t. $\varphi_{2i}(x) > L,\ \ \forall x \in B(x_0, \lambda_L^i)$. Choosing $\lambda_L = \min_{i \notin I_p(x_0)} \lambda_L^i$, it follows for all $i \notin I_p(x_0)$

$$\forall L > 0, \quad \exists \lambda_L > 0 \ \text{ s. t. } \ \varphi_{2i}(x) > L, \quad \forall x \in B(x_0, \lambda_L). \tag{4.4}$$

If $b_i(x_0) > 0$, since the function $b_i$ is continuous, there exists $\delta_i > 0$ such that $b_i(x) > \dfrac{b_i(x_0)}{2} > 0$ on $B(x_0, \delta_i)$. It follows that $\varphi_{2i}(x) = -\dfrac{a_i(x)}{b_i(x)}$ is continuous and bounded on $B(x_0, \delta_i)$. By choosing $\delta = \min_{i \in I_p(x_0)} \delta_i$, we have, for any $i \in I_p(x_0)$, $\varphi_{2i}$ is continuous on $B(x_0, \delta)$, so for all $x \in B(x_0, \delta)$

$$\varphi_{2i}(x) = -\frac{a_i(x)}{b_i(x)} \le L_1 = \max_{j \in I_p(x_0)} \Big( \sup_{y \in B(x_0, \delta)} \varphi_{2j}(y) \Big). \tag{4.5}$$

Now, if $I_p(x_0) = \emptyset$, for any $\varepsilon > 0$, since $\psi_1$ is continuous and positive, take $L_0 = \sup_{y \in B(x_0, \varepsilon)} \psi_1(y)$. Then from (4.4), we can choose $0 < \lambda_{L_0} < \varepsilon$ such that

$$\varphi_{2i}(x) > L_0, \quad \forall x \in B(x_0, \lambda_{L_0}), \quad \forall i \in I.$$

it then follows that

$$\varphi_2(x) = \min_{i \in I} \varphi_{2i}(x) > L_0 \ge \psi_1(x), \quad \forall x \in B(x_0, \lambda_{L_0}),$$

which clearly implies that $w_1(x) = \psi_1(x)$, for all $x \in B(x_0, \lambda_{L_0})$. Thus $w_1$ is continuous at $x_0$. If $I_p(x_0) \neq \emptyset$, combining (4.4) and (4.5), and take $\eta = \min(\lambda_{L_1}, \delta)$ yields that

$$\varphi_{2i}(x) \le L_1 < \varphi_{2j}(x), \quad \forall x \in B(x_0, \eta), \quad \forall i \in I_p(x_0), \quad \forall j \notin I_p(x_0),$$

and consequently

$$\varphi_2(x) = \min_{i \in I} \varphi_{2i}(x) = \min_{i \in I_p(x_0)} \varphi_{2i}(x) \ \ \forall x \in B(x_0, \eta).$$

From this, together with the continuity of $\varphi_{2i}$ on $B(x_0, \eta)$, for all $i \in I_p(x_0)$, it follows that $\varphi_2$ is continuous on $B(x_0, \eta)$. Thus, $w_1 = \min(\psi_1, \varphi_2)$ is continuous on $B(x_0, \eta)$ and then at $x_0$, since $\psi_1$ is continuous on $\mathbb{R}^n$.

Now, we establish the continuity of $w_1$ at the origin in the case of small control property of $V_i$, $i \in I$.

Far each $i \in I$, due to definition of $\varphi_{2i}$ and the small control property of $V_i$, it is not difficult to obtain that

$$\forall \varepsilon > 0, \quad \exists \lambda_i > 0 \ \text{ such that, } \ -\varepsilon < \varphi_{2i}(x), \quad \forall x \in B(0, \lambda_i).$$

Take $\lambda = \min_{i \in I} \lambda_i$, it follows

$$\forall \varepsilon > 0, \quad \exists \lambda > 0 \ \text{ such that, } \ -\varepsilon < \varphi_2(x), \quad \forall x \in B(0, \lambda). \tag{4.6}$$

Continuity of $\psi_1$ at the origin leads to

$$\forall \varepsilon > 0, \quad \exists \gamma > 0 \ \text{ s. t., } \ -\varepsilon < \psi_1(x) < \varepsilon, \quad \forall x \in B(0, \gamma). \tag{4.7}$$

Take $\eta = \min(\lambda, \gamma)$, it follows from (4.6) and (4.7)that for all $x \in B(0, \eta)$

$$\forall \varepsilon > 0, \quad \exists \eta > 0 \ \text{ s. t., } \ -\varepsilon < w_1(x) = \min(\psi_1(x), \varphi_2(x)) < \varepsilon.$$

That is, $w_1$ is continuous at the origin and so on $\mathbb{R}^n$.
The continuity of $w_2$ can be treated similarly and is omitted here. Thus $u$ is continuous on $\mathbb{R}^n$.
**Global asymptotic stability:** Due to the continuity of $u$, we deduce that for each $i \in I$, the coefficients of the closed-loop system: $dx = f_i(x)dt + u(x)g_i(x)dt + h_i(x)dw$, and the associated infinitesimal operator $\mathcal{L}_u V_i(x) = a_i(x) + b_i(x)u(x)$ are continuous on $\mathbb{R}^n$. In view of Lemma 2.1, we only have to verify that for any $i \in I$,

$$\mathcal{L}_u V_i(x) = a_i(x) + b_i(x)u(x) < 0, \quad \forall x \in \mathbb{R}^n \setminus \{0\}, \tag{4.8}$$

according to the sign of $b_i(x)$, $x \neq 0$.

- If $b_i(x) > 0$. On one hand, From the definitions of $w_1$, $\varphi_2$ and $\varphi_{2i}$, we obtain

$$w_1(x) \leq \varphi_2(x) \leq \varphi_{2i}(x) = -\frac{a_i(x)}{b_i(x)}. \tag{4.9}$$

  On the other hand, definitions of $\psi_2$ and $\psi_{2i}$ give, $\psi_2(x) \leq \psi_{2i}(x) = k_i(x) < -\frac{a_i(x)}{b_i(x)}$. From Assumption 4.1, we get $\varphi_1(x) < \varphi_2(x) \leq -\frac{a_i(x)}{b_i(x)}$. Hence,

$$w_2(x) = \max(\psi_2(x), \varphi_1(x)) < -\frac{a_i(x)}{b_i(x)}. \tag{4.10}$$

  (4.9) and (4.10) lead to

$$u(x) = \frac{w_1(x) + w_2(x)}{2} < -\frac{a_i(x)}{b_i(x)}.$$

  Thus, $\mathcal{L}_u V_i(x) = a_i(x) + b_i(x)u(x) < 0$.

- If $b_i(x) < 0$. Similarly to the previous case, on one hand, from definitions of $w_2$, $\varphi_1$ and $\varphi_{1i}$, we have

$$w_2(x) \geq \varphi_1(x) \geq \varphi_{1i}(x) = -\frac{a_i(x)}{b_i(x)}. \tag{4.11}$$

  On the other hand, we have, $\psi_1(x) \geq \psi_{1i}(x) = k_i(x) > -\frac{a_i(x)}{b_i(x)}$. By Assumption 4.1, $\varphi_2(x) > \varphi_1(x) \geq -\frac{a_i(x)}{b_i(x)}$ and then

$$w_1(x) = \min(\psi_1(x), \varphi_2(x)) > -\frac{a_i(x)}{b_i(x)}. \tag{4.12}$$

  Thus, $\mathcal{L}_u V_i(x) = a_i(x) + b_i(x)u(x) < 0$, since (4.11) and (4.12) yield

$$u(x) = \frac{w_1(x) + w_2(x)}{2} > -\frac{a_i(x)}{b_i(x)}.$$

- If $b_i(x) = 0$, then $\mathcal{L}_u V_i(x) = a_i(x) < 0$, since $V_i$ is a SCLF for the system $S_i$.

Thus, $\mathcal{L}_u V_i$ is negative definite as desired.
Finally , $\mathcal{L}_u V_i$ is continuous and negative definite, and since $V_i$ is a SCLF for system $S_i$, $i \in I$, that is, $V_i$ is proper and positive definite, according to Lemma 4.3 in [17], there exit class $\mathcal{K}_\infty$ functions $\alpha$ and $\beta$ and class $\mathcal{K}$ function $\xi$, defined on $\mathbb{R}^+$, such that

$$\alpha(\|x\|) \leq V(x) \leq \beta(\|x\|), \quad \forall x \in \mathbb{R}^n,$$

and

$$\mathcal{L}_u V_i(x) = a_i(x) + b_i(x)u(x) \leq -\xi(\|x\|), \quad \forall x \in \mathbb{R}^n.$$

The conclusion follows with help of Lemma 2.1 and this completes the proof of Theorem 4.3. $\square$

### 4.3 Simultaneous stabilization via bounded control

This section will show that, under appropriate assumption on SCLFs, simultaneous stabilization can be guaranteed for the collection of systems in (3.1) by bounded feedback.
Here, we assume that there exists a SCLF $V$ for system (3.2) satisfying the small control property and with controls in $\mathcal{B}_1 = \{u \in \mathbb{R} \ | \ -1 < u < 1\}$, that is

$$\inf_{u\in\mathcal{B}_1} \mathcal{L}_u V(x) = \inf_{u\in\mathcal{B}_1} \Big(a(x) + b(x)u\Big) < 0, \quad \forall x \in \mathbb{R}^n \setminus \{0\}. \tag{4.13}$$

Let (as defined by Lin-Sontag's universal formula [25])

$$p(x) = \begin{cases} -\dfrac{a(x) + \sqrt{a^2(x) + b^4(x)}}{b(x)\Big(1 + \sqrt{1 + b^2(x)}\Big)}, & \text{if } \ b(x) \neq 0, \\ 0, & \text{if } \ b(x) = 0, \end{cases} \tag{4.14}$$

where the function $a$ and $b$ are defined in (3.3).

**Theorem 4.4.** *For system (3.2), suppose that there exists a SCLF $V$ with controls in $\mathcal{B}_1$, satisfying the small control property, Then, the origin solution to (3.2) is globally asymptotically stable in probability with the continuous feedback $u = p(x)$ defined in (4.14) which takes values in $\mathcal{B}_1$.*

*Proof.* For the continuity of the feedback $p$ and the global asymptotic stability of the resulting closed-loop system, the proof is similar to that of Theorem 4.1.
For the boundedness of the feedback $p$, let us define (as it has been done in [25]) a function $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$\phi(a, b) = \begin{cases} -\dfrac{a + \sqrt{a^2 + b^4}}{b\Big(1 + \sqrt{1 + b^2}\Big)}, & \text{if } \ b \neq 0, \\ 0, & \text{if } \ b = 0, \end{cases} \tag{4.15}$$

and let $\mathcal{D} = \{(a, b) \in \mathbb{R}^2 \mid a < |b|\}$.
Now, note that property (4.13) is equivalent to $a(x) < |b(x)|$, for all $x \in \mathbb{R}^n \setminus \{0\}$, that is $(a(x), b(x)) \in \mathcal{D}$ for all $x \in \mathbb{R}^n \setminus \{0\}$. By Lemma 2.3 of [25] we have $|\phi(a, b)| < 1$ for all $(a, b) \in \mathcal{D}$. Hence $|p(x)| = |\phi(a(x), b(x))| < 1$ which concludes the proof of the Theorem 4.4. □

To derive simultaneously stabilizing controllers, for each $i \in I$, assume that $V_i$ is a SCLF of system $S_i$, satisfying the small control property and takes values in $\mathcal{B}_1$. Define

$$p_i(x) = \begin{cases} -\dfrac{a_i(x) + \sqrt{a_i^2(x) + b_i^4(x)}}{b_i(x)\Big(1 + \sqrt{1 + b_i^2(x)}\Big)}, & \text{if } \ b_i(x) \neq 0, \\ 0, & \text{if } \ b_i(x) = 0 \end{cases} \tag{4.16}$$

where $a_i$ and $b_i$ are as defined in (4.2). Therefore, according to Theorem 4.4, the functions $p_i$, $i \in I$, is continuous, takes values in $\mathcal{B}_1$ and globally asymptotically stabilizes system $S_i$.
In the following, we replace the function $k_i$ by the function $p_i$ in the definitions of $\psi_{1i}$ and $\psi_{2i}$.

**Remark 4.6.** *As in Remark 4.2, for each $i \in I$, since $a_i(x) + \sqrt{a_i^2(x) + b_i^4(x)} > 0$, it folows that $b_i(x)p_i(x) < 0$, $i\,f\, b_i(x) \neq 0$. So, taking into account that $-1 < p_i(x) < 1$, for all $x \in \mathbb{R}^n$, we have*

$$\begin{aligned} &0 < p_i(x), \quad \textit{if} \ \ b_i(x) < 0, \quad \textit{and then} \quad 0 \le \psi_1(x) < 1, \\ &\textit{and}, \\ &p_i(x) < 0, \quad \textit{if} \ \ b_i(x) > 0, \quad \textit{and then} \quad -1 < \psi_2(x) \le 0. \end{aligned}$$

Now we can state the following theorem.

**Theorem 4.5.** *Consider the collection of systems in (3.1). If there exists a SCLF $V_i$ with control in $\mathcal{B}_1$, that satisfies the small control property, for every $S_i$, $i \in I$, and Assumption 4.1 holds, then the feedback*

$$u(x) = \frac{w_1(x) + w_2(x)}{2} \tag{4.17}$$

*is continuous, globally asymptotically stabilizes in probability the collection of systems in (3.1) simultaneously and takes values in $\mathcal{B}_1$.*

*Proof.* First, quite similarly to the proof of Theorem 4.3, we can prove that the feedback (4.17) is continuous and globally asymptotically stabilizes systems $S_i$, $i \in I$.
Second, we still have to verify that the feedback (4.17) takes values in $\mathcal{B}_1$. To do this, we shall verify that for all $x \in \mathbb{R}^n$, $-1 < w_1(x) < 1$ according to the set $I_p(x)$.

- If $I_p(x) = \emptyset$, then $w_1(x) = \psi_1(x)$. From this together with the fact that we replaced $k_i$ by $p_i$ in the definition of $\psi_{1i}$, we get $0 \leq w_1(x) < 1$ according to Remark 4.6.

- If $I_p(x) \neq \emptyset$, from the definition of $\varphi_2$, let $i_0 \in I_p(x)$ such that $\varphi_2(x) = -\dfrac{a_{i_0}(x)}{b_{i_0}(x)}$. Since $p_{i_0}(x) \in \mathcal{B}_1$ and $b_{i_0}(x) > 0$, we have

$$-1 < u_{i_0}(x) < -\frac{a_{i_0}(x)}{b_{i_0}(x)} = \varphi_2(x).$$

  This last inequality and Remark 4.6 lead to

$$-1 < \min(\psi_1(x), \varphi_2(x)) = w_1(x) \leq \psi_1(x) < 1.$$

Thus, $w_1(x) \in \mathcal{B}_1$ for all $x \in \mathbb{R}^n$. The reasoning is similar to prove that $w_2$ takes values in $\mathcal{B}_1$, then so is for $u$ and this completes the proof of Theorem 4.5. □

# 5 Some illustrative examples

In this section, we present two examples to illustrate the effectiveness of the results obtained in this paper to design a simultaneously stabilizing feedback for some nonlinear stochastic systems.

## Example 1

Consider a system with the following three possible modes

$$S_1 : \begin{cases} dx_1 = -2x_1^2 x_2^2 dt + x_1 u dt \\ dx_2 = (-x_2 - 2x_1 x_2^3) dt + x_2 u dt + x_2 d\omega \end{cases}$$

$$S_2 : \begin{cases} dx_1 = -x_1 dt - u dt + 2x_1 x_2 d\omega \\ dx_2 = 2x_1 x_2^3 dt - x_2 u dt \end{cases}$$

$$S_3 : \begin{cases} dx_1 = -x_1 dt - u dt + 2x_1 x_2 d\omega \\ dx_2 = -x_2 dt + x_2 d\omega \end{cases}$$

We want to find a continuous function $p : \mathbb{R}^2 \to \mathbb{R}^2$ such that the state feedback controller $u(x) = p(x)$ globally asymptotically stabilizes in probability the three systems $S_1$, $S_2$ and $S_3$. To this end, let

$$V_1(x) = \frac{x_1^2 + x_2^2}{2}, \quad V_2(x) = \frac{x_1^2}{2} + \frac{x_2^4}{4} \quad \text{and} \quad V_3(x) = \frac{x_1^2 + x_2^2}{2}.$$

Denote by $\mathcal{L}_u V_1$, $\mathcal{L}_u V_2$ and $\mathcal{L}_u V_3$ the infinitesimal operator associated with system $S_1$, $S_2$ and $S_3$, respectively, we have

$$\begin{aligned}
\mathcal{L}_u V_1(x) &= -\tfrac{1}{2}x_2^2 - 2x_1x_2^2(x_1^2 + x_2^2) + (x_1^2 + x_2^2)u, \\
\mathcal{L}_u V_2(x) &= -x_1^2 + 2x_1x_2^2(x_1 + x_2^4) - (x_1 + x_2^4)u, \\
\mathcal{L}_u V_3(x) &= -x_1^2 - \tfrac{x_2^2}{2} + 2x_1^2x_2^2 - x_1u,
\end{aligned}$$

that is

$$\begin{aligned}
a_1(x) &= -\tfrac{1}{2}x_2^2 - 2x_1x_2^2(x_1^2 + x_2^2), \quad \text{and} \quad b_1(x) = x_1^2 + x_2^2, \\
a_2(x) &= -x_1^2 + 2x_1x_2^2(x_1 + x_2^4), \quad \text{and} \quad b_2(x) = -(x_1 + x_2^4), \\
a_3(x) &= -x_1^2 - \tfrac{x_2^2}{2} + 2x_1^2x_2^2, \quad \text{and} \quad b_3(x) = -x_1.
\end{aligned}$$

We can verify that $V_1$, $V_2$ and $V_3$ are SCLFs for system $S_1$, $S_2$ and $S_3$, respectively, satisfying the small control property. Define

$$\begin{aligned}
\mathcal{A}_1 &= \{x \in \mathbb{R}^n \mid x_1 < -x_2^4\} \\
\mathcal{A}_2 &= \{x \in \mathbb{R}^n \mid x_1 = -x_2^4 \ \text{and} \ x_1 \neq 0\} \\
\mathcal{A}_3 &= \{x \in \mathbb{R}^n \mid -x_2^4 < x_1 < 0\} \\
\mathcal{A}_4 &= \{x \in \mathbb{R}^n \mid x_1 = 0 \ \text{and} \ x_2 \neq 0\} \\
\mathcal{A}_5 &= \{x \in \mathbb{R}^n \mid 0 < x_1\}
\end{aligned}$$

Clearly, $\mathbb{R}^n \setminus \{0\} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3 \cup \mathcal{A}_4 \cup \mathcal{A}_5$. Moreover, $I_p(x) = \{1,2,3\}$ and $I_n(x) = \emptyset$ if $x \in \mathcal{A}_1$, $I_p(x) = \{1,3\}$ and $I_n(x) = \emptyset$ if $x \in \mathcal{A}_2$, $I_p(x) = \{1,3\}$ and $I_n(x) = \{2\}$ if $x \in \mathcal{A}_3$, $I_p(x) = \{1\}$ and $I_n(x) = \{2\}$ if $x \in \mathcal{A}_4$, $I_p(x) = \{1\}$ and $I_n(x) = \{2,3\}$ if $x \in \mathcal{A}_5$.
Now, if $x \in \mathcal{A}_3 \cup \mathcal{A}_4 \cup \mathcal{A}_5$, then

$$\frac{a_1(x)}{b_1(x)} - \frac{a_2(x)}{b_2(x)} = -\frac{x_2^2}{2(x_1^2 + x_2^2)} - \frac{x_1^2}{x_1 + x_2^4} < 0.$$

If $x \in \mathcal{A}_3$, then $\dfrac{a_3(x)}{b_3(x)} - \dfrac{a_2(x)}{b_2(x)} = x_1 + \dfrac{x_2^2}{2x_1} - \dfrac{x_1^2}{x_1 + x_2^4} < 0.$
If $x \in \mathcal{A}_5$, then $\dfrac{a_1(x)}{b_1(x)} - \dfrac{a_3(x)}{b_3(x)} = -\dfrac{x_2^2}{2(x_1^2 + x_2^2)} - x_1 - \dfrac{x_2^2}{2x_1} < 0$. That is, Assumption 4.1, i.e. $\varphi_1(x) < \varphi_2(x)$ for all $x \in \mathbb{R}^n$, holds since $I_n(x) = \emptyset$ and then, $\varphi_1(x) = -\infty$ for all $x \in \mathcal{A}_1 \cup \mathcal{A}_2 \cup \{0\}$. Thus, thanks to Theorem 4.3, the feedback

$$u(x) = \frac{w_1(x) + w_2(x)}{2}, \tag{5.1}$$

is continuous on $\mathbb{R}^2$ and globally asymptotically stabilizes in probability the systems $S_1$, $S_2$ and $S_3$ simultaneously, with for each $i \in \{1,2,3\}$

$$u_i(x) = \begin{cases} -\dfrac{a_i(x) + \sqrt{a_i^2(x) + b_i^4(x)}}{b_i(x)}, & \text{if } b_i(x) \neq 0, \\ 0, & \text{if } b_i(x) = 0, \end{cases}$$

and

$$\left.\begin{aligned} w_1(x) &= \min\Big(0, \min_{i=1,2,3}(-\frac{a_i(x)}{b_i(x)})\Big), \\ w_2(x) &= \min_{i=1,2,3}(u_i(x)), \end{aligned}\right. \quad \text{if} \quad x \in \mathcal{A}_1.$$

$$\left.\begin{aligned} w_1(x) &= \min\Big(0, \min_{i=1,3}(-\frac{a_i(x)}{b_i(x)})\Big), \\ w_2(x) &= \min_{i=1,3}(u_i(x)), \end{aligned}\right. \quad \text{if} \quad x \in \mathcal{A}_2.$$

$$\left.\begin{aligned} w_1(x) &= \min\Big(u_2(x), \min_{i=1,3}(-\frac{a_i(x)}{b_i(x)})\Big), \\ w_2(x) &= \max\Big(\min_{i=1,3}(u_i(x)), -\frac{a_2(x)}{b_2(x)}\Big), \end{aligned}\right. \quad \text{if} \quad x \in \mathcal{A}_3.$$

$$\left.\begin{aligned} w_1(x) &= \min\Big(u_2(x), -\frac{a_1(x)}{b_1(x)}\Big), \\ w_2(x) &= \max\Big(u_1(x), -\frac{a_2(x)}{b_2(x)}\Big), \end{aligned}\right. \quad \text{if} \quad x \in \mathcal{A}_4.$$

$$\left.\begin{aligned} w_1(x) &= \min\Big(\max_{i=2,3}(u_i(x)), -\frac{a_1(x)}{b_1(x)})\Big), \\ w_2(x) &= \max\Big(u_1(x), \max_{i=2,3}(-\frac{a_i(x)}{b_i(x)})\Big), \end{aligned}\right. \quad \text{if} \quad x \in \mathcal{A}_5.$$

## Example 2

Consider a system with the following two possible modes

$$S_1' : \begin{cases} dx_1 = (x_2^3 - 2\dfrac{(x_1^2+x_2^2)}{1+x_1^2})dt + (-2x_1+x_2)udt \\ dx_2 = (\dfrac{x_1^2+x_2^2}{1+x_1^2} - x_2^3)dt + (x_1-x_2)udt + \frac{1}{2}x_2^2 d\omega \end{cases}$$

$$S_2' : \begin{cases} dx_1 = (-x_1 + \dfrac{x_1^2+x_2^2}{1+x_1^2})dt + x_1 udt + \sqrt{2}\dfrac{x_1}{\sqrt{1+x_1^2}}d\omega \\ dx_2 = -\dfrac{x_1^2 x_2}{1+x_1^2}dt + x_2 udt \end{cases}$$

Taking the following functions $V_1(x) = \dfrac{(x_1+x_2)^2}{2} + \dfrac{x_2^2}{2}$ and $V_2(x) = \dfrac{x_1^2+x_2^2}{2}$. Let $\mathcal{L}_u V_1$ and $\mathcal{L}_u V_2$ the infinitesimal operator associated with system $S_1$ and $S_2$, respectively, we have $\mathcal{L}_u V_1(x) = a_1(x) + b_1(x)u$ and $\mathcal{L}_u V_2(x) = a_2(x) + b_2(x)u$, with $a_1(x) = -\frac{3}{4}x_2^4 - \dfrac{x_1}{1+x_1^2}(x_1^2+x_2^2)$, $\quad b_1(x) = -x_1^2 - x_2^2$, $a_2(x) = \dfrac{(-x_1^2+x_1)(x_1^2+x_2^2)}{1+x_1^2}$, $\quad b_2(x) = x_1^2+x_2^2$. It can be seen that $V_i$ is a SCLF for system $S_i$ $(i = 1,2)$. It is easy to verify that both $V_1$ and $V_2$ satisfy the the small control property (SCP). On the other hand, straightforward calculations lead to

$$\frac{a_1(x)}{b_1(x)} = \frac{3x_2^4}{4(x_1^2+x_2^2)} + \frac{x_1}{1+x_1^2} \quad \text{and} \quad \frac{a_2(x)}{b_2(x)} = \frac{-x_1^2+x_1}{1+x_1^2},$$

then, $-\dfrac{a_1(x)}{b_1(x)} < 1$ and $-1 < -\dfrac{a_2(x)}{b_2(x)}$, for all $x \in \mathbb{R}^2 \setminus \{0\}$, and, since $b_1(x) < 0$, $b_2(x) > 0$, for all $x \neq 0$, it follows

$$\inf_{u \in \mathcal{B}_1} \mathcal{L}_u V_1(x) < 0, \quad \text{and} \quad \inf_{u \in \mathcal{B}_1} \mathcal{L}_u V_2(x) < 0, \quad \forall x \in \mathbb{R}^2 \setminus \{0\}.$$

Notice that the stabilizability condition: $b_1(x)$ and $b_2(x)$ have the same sign for all $x \in \mathbb{R}^2$, stated in [13] is not satisfied, since $b_1(x) < 0$, and $b_2(x) > 0$, $x \neq 0$. However, we have $\frac{a_2(x)}{b_2(x)} - \frac{a_1(x)}{b_1(x)} = -\frac{x_1^2}{1+x_1^2} - \frac{3x_2^4}{4(x_1^2+x_2^2)} < 0, \quad \forall x \in \mathbb{R}^2 \setminus \{0\}$, that is, Assumption 4.1, i.e. $\varphi_1(x) < \varphi_2(x)$ for all $x \in \mathbb{R}^2$, is satisfied. Thus, according to Theorem 4.5, the feedback

$$u(x) = \frac{w_1(x) + w_2(x)}{2}, \tag{5.2}$$

is continuous on $\mathbb{R}^2$, takes values in $\mathcal{B}_1$ and globally asymptotically stabilizes in probability systems $S_1$ and $S_2$ simultaneously, with, using (4.16) and notations in Theorem 4.5

$$w_1(x) = \begin{cases} \min\left( -\dfrac{a_1(x) + \sqrt{a_1^2(x) + b_1^4(x)}}{b_1(x)\left(1 + \sqrt{1 + b_1^2(x)}\right)}, -\dfrac{a_2(x)}{b_2(x)} \right) & \text{if } \ x \neq 0, \\ 0, & \text{if } \ x = 0, \end{cases}$$

and

$$w_2(x) = \begin{cases} \max\left( -\dfrac{a_2(x) + \sqrt{a_2^2(x) + b_2^4(x)}}{b_2(x)\left(1 + \sqrt{1 + b_2^2(x)}\right)}, -\dfrac{a_1(x)}{b_1(x)} \right) & \text{if } \ x \neq 0, \\ 0, & \text{if } \ x = 0. \end{cases}$$

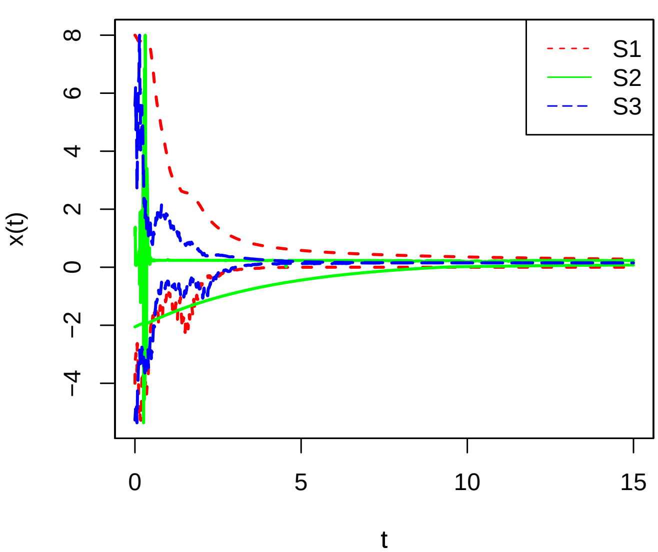


Figure 1: The responses of the closed-loop systems S1, S2 and S3 with the same feedback (5.1).

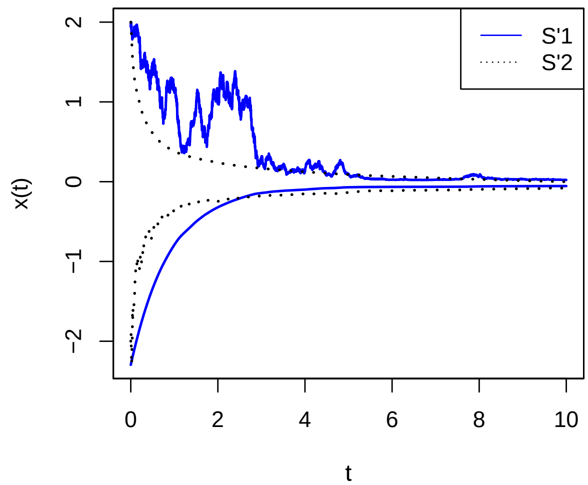


Figure 2: The responses of the closed-loop systems S'1 and S'2 with the same feedback (5.2).

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